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Free convection from a vertical cooling fibre

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Abstract. We study the free-convective boundary-layer flow that is induced when a slender circular cylinder emerges from an orifice and moves vertically downwards. We demonstrate, by numerical solution of the equations, that the boundary-layer solution develops a singularity at a finite point, where the limiting form of solution is as predicted by Kuiken [3] for an analogous two-dimensional flow.

1. Introduction

There are many industrial processes which involve the cooling of cylinders, fibres or sheets of material. An example is provided by the manufacture of glass fibre. The fibre issues from an orifice at the bottom of a crucible containing liquid glass. The fibre cools as it moves downwards, before being rolled onto a drum. It is the free-convective flow associated with such a process that we are concerned with in this paper.

Kuiken [1, 2, 3] has been much concerned with cooling in such extrusion processes. In [1, 2] he considered the cooling of thin sheets and slender cylinders in the absence of free-convective effects. However, in a study [3] of the cooling of a hot thin sheet moving vertically downwards he included free convection due to gravity and discovered a most interesting singular solution, of self-similar form, of the governing equations. This solution of the boundary-layer equations has the property that it decays algebraically, rather than exponentially, far from the sheet. On that account, and bearing in mind the work of Goldstein [4], and Brown and Stewartson [5] on boundary layers with algebraic decay, the role of Kuiken's solution might be thought to be that of a limit solution as the singular point is approached. However Khan and Stewartson [6], from consideration of a full numerical solution of the governing parabolic partial differential equations, show that for this problem the similarity solution gives a remarkably accurate estimate of flow properties at the sheet over a much greater extent of it than might have been presupposed.

In the present paper we study the free-convective boundary layer on a heated, downward moving circular cylinder which is sufficiently slender to model a fibre. Although there is no analogue of Kuiken's self-similar solution [3] for this problem, we demonstrate that it is the leading term in a series solution that may be developed about a singular point. A full numerical solution of the governing boundary-layer equations identifies the point at which the solution breaks down, and at the same time confirms Kuiken's result as the limiting form of the solution at that point.

2. Problem formulation

We consider the axisymmetric, laminar, free-convective flow about a thin, hot vertical cylinder or fibre of circular cross-section and diameter $2a$. The cylinder moves with speed u_s

in the direction of the gravity vector **g**, emerging from an orifice with excess temperature T_e over the ambient temperature T_∞, and disappearing into another orifice with an excess temperature $T_1 < T_e$ which is unknown *a priori*. The exposed length of the cylinder is L. With reference to Fig. 1 we choose cylindrical polar co-ordinates (x', r'), with origin at the lower orifice, and corresponding velocity components (u', v'). We write the excess temperature as $T' + T_\infty$. If we assume that the thickness of the moving region of fluid adjacent to the cylinder is small compared to L, then the boundary-layer equations expressing conservation of mass, momentum and energy, for the problem we have defined, are

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{v'}{a+y'} = 0\,, \tag{2.1}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = \nu\left(\frac{\partial^2 u'}{\partial y'^2} + \frac{1}{a+y'}\frac{\partial u'}{\partial y'}\right) + g\beta T'\,, \tag{2.2}$$

$$u'\frac{\partial T'}{\partial x'} + v'\frac{\partial T'}{\partial y'} = \frac{\nu}{\sigma}\left(\frac{\partial^2 T'}{\partial y'^2} + \frac{1}{a+y'}\frac{\partial T'}{\partial y'}\right) \tag{2.3}$$

where $y' = r' - a$, ν is the kinematic viscosity of the fluid and β its coefficient of thermal expansion; σ is the Prandtl number. We suppose that the temperature of the moving cylinder, or fibre, is constant across its cross-section with excess value $T_s(x')$. The boundary

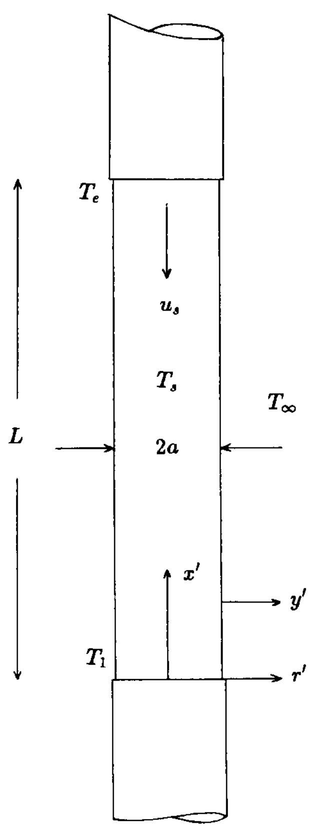

Fig. 1. Definition sketch.

conditions for our equations (2.1) to (2.3) are, then,

$$\left.\begin{aligned} &u' = -u_s\,, \quad v' = 0\,, \quad T' = T_s(x')\,; \quad y' = 0\,, \quad x' \geqslant 0\,, \\ &u',\, T' \to 0 \text{ as } y' \to \infty\,, \quad x' \geqslant 0\,, \\ &u',\, T' = 0\,; \quad x' = 0\,, \quad y' > 0\,. \end{aligned}\right\} \tag{2.4}$$

Since the cylinder temperature $T_s(x')$ is not known *a priori* an additional condition is required. This is determined from a consideration of the heat balance in the cylinder itself. Between two fixed stations the net flux of heat carried into the section, by the cylinder's own motion, is balanced by a transport of heat across the curved surface of the cylinder by conduction. This may be expressed mathematically as

$$\frac{\partial T'}{\partial x'} + \lambda \frac{\partial T'}{\partial y'} = 0 \quad \text{at } y' = 0\,, \tag{2.5}$$

with $\lambda = 2\rho c\nu/\sigma a \rho_s c_s u_s$, where ρ is the density, c the specific heat and a suffix s refers to the solid material. In (2.5) $\partial T'/\partial y'|_{y'} = 0$ is calculated from the solution of (2.3).

We now introduce dimensionless variables by writing $x' = lx$, $y' = ay$, $u' = U'u$, $v' = \varepsilon U'v$, $T' = T_1 T$. Here $l = O(L)$ is an axial length scale, $\varepsilon = a/l \ll 1$, and for the velocity scale we choose $U' = (g\beta T_1 l)^{1/2}$ for this free-convection dominated problem. The equations (2.1) to (2.3) and the condition (2.5) may then be written in non-dimensional form, respectively, as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{1+y} = 0\,,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\varepsilon\,\mathrm{Gr}^{1/2}}\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{1+y}\frac{\partial u}{\partial y}\right) + T\,,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\varepsilon\,\mathrm{Gr}^{1/2}\sigma}\left(\frac{\partial^2 T}{\partial y^2} + \frac{1}{1+y}\frac{\partial T}{\partial y}\right),$$

$$\frac{\partial T}{\partial x} + \frac{\lambda}{\varepsilon}\frac{\partial T}{\partial y} = 0\,,$$

where $\mathrm{Gr} = g\beta T_1 l a^2/\nu^2$ is a Grashof number, with $\mathrm{Gr} \gg 1$ for consistency with our framework of boundary-layer theory. Unlike the two-dimensional situation considered by Khan and Stewartson [6], and the two-dimensional equations may be derived from (2.1) to (2.3) by formally allowing $a \to \infty$, our non-dimensional equations do not appear to be reducible to a suitable canonical form. For the purposes of the present paper, as outlined in Section 1, we proceed as follows. We make the approximation of unit Prandtl number, which is acceptable for air. Also, since in our formulation $\varepsilon \ll 1$ and $\mathrm{Gr} \gg 1$ we take $\varepsilon\,\mathrm{Gr}^{1/2} = 1$. Further since $\lambda \ll 1$, and indeed is required to be so if our boundary-layer theory is to apply as we see from (2.5), we also take $\lambda/\varepsilon = 1$. The values we have taken are representative of a practical situation, and the results we obtain below will only be modified in detail if other values for these parameters are chosen. We make one further simplification which is that $u_s \ll U'$; that is we assume the fibre's downward velocity to be small compared with the induced free-convective velocity in the surrounding medium. This assumption will be justified in practice except close to the origin $x = 0$.

With the above assumptions the problem we address is as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{1+y}=0\,, \tag{2.6}$$

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+\frac{1}{1+y}\,\frac{\partial u}{\partial y}+T\,, \tag{2.7}$$

$$u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}=\frac{\partial^2 T}{\partial y^2}+\frac{1}{1+y}\,\frac{\partial T}{\partial y}\,, \tag{2.8}$$

together with

$$\left.\begin{aligned}&u=v=0\,,\quad \frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}=0 \quad \text{at } y=0\,,\quad x>0\,,\\ &u\to 0\,,\quad T\to 0 \quad \text{as } y\to\infty \text{ for } x>0\,,\\ &u=0\,,\quad T=1 \quad \text{at } x=0\,,\quad y=0\,,\\ &u=T=0 \quad \text{at } x=0\,,\quad y>0\,.\end{aligned}\right. \tag{2.9}$$

We remark finally that, following the solution of (2.6) to (2.9), the temperature T_1 of the fibre at $x=0$ is determined in terms of the exit temperature T_e at $x_e=L/l$ as $T_1=T_e/T(x_e,0)$.

3. Numerical results

For the analogous problem of the cooling of a vertical two-dimensional thin film Kuiken [3] has obtained a remarkable similarity solution of the equations analogous to (2.6) to (2.9). The solution is remarkable in that it is singular at a finite point $x=x_0>0$ on the film. As that point is approached the temperature and vertical velocity increase without bound, with the thickness of the boundary layer diminishing to zero. For the problem posed by equations (2.6) to (2.9) there appears to be no similarity solution. However, the solution of Kuiken can be used as the leading term in the development of a singular solution close to the point $x=x_0$. Thus, we write

$$\left.\begin{aligned}u&=\frac{\mu^{1/2}}{x_0-x}\sum_{n=0}^{\infty}(x_0-x)^n f_n'(\eta)\,,\\ v&=\frac{\mu^{1/4}}{x_0-x}\sum_{n=0}^{\infty}(x_0-x)^n(nf_n-\eta f_n')\,,\\ T&=\frac{\mu}{(x_0-x)^3}\sum_{n=0}^{\infty}(x_0-x)^n\theta_n(\eta)\,,\end{aligned}\right\} \tag{3.1}$$

where

$$\eta=\frac{\mu^{1/4}y}{x_0-x}\,, \tag{3.2}$$

and μ is a constant to be determined. The leading terms of (3.1) provide the Kuiken similarity solution, with f_0, θ_0 satisfying

$$f_0''' - f_0'^2 + \theta_0 = 0 , \qquad \theta_0'' - 3\sigma f_0'\theta_0 = 0 , \tag{3.3}$$

where a prime denotes differentiation with respect to η. Kuiken solved these equations numerically subject to the conditions $f_0(0) = f_0'(0) = 0$, $\theta_0(0) = 1$, $\theta_0(\infty) = f_0'(\infty) = 0$ and obtained, in particular, for $\sigma = 1$ the results

$$f_0''(0) = 0.69321 , \qquad \theta_0'(0) = -0.76986 , \qquad f_0(\infty) = 2.43998 . \tag{3.4}$$

The constant μ in (3.1), (3.2) is then obtained from the heat flow condition in (2.9) as

$$\mu^{1/4} = \{-\tfrac{1}{3}\theta'(0)\}^{-1} \quad \text{or} \quad \mu = 2.30589 \times 10^2 . \tag{3.5}$$

A feature of the solution of (3.3), noted by Kuiken is that the decay of f_0', θ_0 to zero as $\eta \to \infty$ is algebraic rather than exponential. Such behaviour of solutions of the boundary-layer equations had been observed by Goldstein [4] who cast doubt upon their value, since solutions which exhibit such behaviour over a finite length cannot be satisfactorily matched to an inviscid potential flow. However Brown and Stewartson [5] subsequently showed that algebraic decay is usually only encountered at one streamwise point, and is associated with non-commutative limits. As a consequence Kuiken conjectured that his solution should perhaps be interpreted as a limit solution of the boundary-layer equations as $x \to x_0$. This conjecture provided the starting point for the investigation by Khan and Stewartson [6]. They not only verified Kuiken's conjecture, with $x_0 = 6.01252$, but in addition showed that the similarity solution predicts conditions close to the boundary very accurately for $x \geqslant 2$. Less good is a comparison with the displacement thickness which represents the overall structure of the boundary layer and its previous history. We may conclude that the similarity solution is indeed a limit solution in the inner part of the boundary layer, as originally conceived by Kuiken, but that it has a wider range of validity than might have been supposed, since the situation in the inner part of the layer tends to be dominated by local conditions.

For the axisymmetric problem under consideration there is no comparable similarity solution but, as we have already indicated in (3.1), (3.2), the Kuiken solution appears to dominate the flow as a singular point x_0 is approached. The principal aim of the present paper is to demonstrate that this is indeed the case. To achieve this aim we integrate our governing equations (2.6) to (2.9) from $x = 0$ in the direction of x increasing, until a singular behaviour is encountered. To enable this we introduce new co-ordinates which are sympathetic, not only to the nature of the singularity as $x \to x_0$ but also to the essential singularity at $x = 0$. Close to $x = 0$ the free-convective boundary layer behaves like that on a heated semi-infinite plate with a thickness $O(x^{1/4})$, and streamwise velocity $O(x^{1/2})$. Thus we introduce co-ordinates X, Y with

$$x = \frac{3}{4} X^{4/3} , \qquad y = \frac{X^{1/3}(X_0 - X)}{X_0} Y , \tag{3.6}$$

where X_0, corresponding to x_0, remains to be determined. The velocity components and temperature are then transformed as

$$u = \frac{X_0 X^{2/3}}{X_0 - X} U(X, Y), \qquad V = \frac{X_0}{X^{1/3}(X_0 - X)} V(X, Y), \qquad T = \frac{X_0^3}{(X_0 - X)^3} \Theta(X, Y). \tag{3.7a,b,c}$$

From (2.6) to (2.9), (3.6) and (3.7) we then have, with

$$\left. \begin{aligned} &F_1(X) = \frac{2X_0 + X}{3X_0}, \qquad F_2(X) = \frac{X(X_0 - X)}{X_0}, \qquad F_3(X, Y) = \left(\frac{4X - X_0}{3X_0}\right) Y, \\ &F_4(X, Y) = \frac{X^{1/3}(X_0 - X)}{X_0 + X^{1/3}(X_0 - X) Y}, \qquad F_5(X) = \frac{3X}{X_0}, \end{aligned} \right\} \tag{3.8}$$

$$F_2 \frac{\partial U}{\partial X} + F_3 \frac{\partial U}{\partial Y} + F_1 U + \frac{\partial V}{\partial Y} + F_4 V = 0, \tag{3.9}$$

$$\frac{\partial^2 U}{\partial Y^2} - F_2 U \frac{\partial U}{\partial X} + (F_4 - F_3 U - V) \frac{\partial U}{\partial Y} - F_1 U^2 + \Theta = 0, \tag{3.10}$$

$$\frac{\partial^2 \Theta}{\partial Y^2} - F_2 U \frac{\partial \Theta}{\partial X} + (F_4 - F_3 U - V) \frac{\partial \Theta}{\partial Y} - F_5 U \Theta = 0, \tag{3.11}$$

together with

$$\left. \begin{aligned} &U = V = 0, \qquad 3\Theta + (X_0 - X) \frac{\partial \Theta}{\partial X} + X_0 \frac{\partial \Theta}{\partial Y} = 0; \qquad Y = 0, \quad X > 0, \\ &U \to 0, \quad \Theta \to 0; \quad Y \to \infty, \quad X > 0, \\ &U = 0, \quad \Theta = 1; \quad X = Y = 0, \\ &U = \Theta = 0; \quad X = 0, \quad Y > 0. \end{aligned} \right\} \tag{3.12}$$

If we set $X \equiv 0$ then equations (3.9) to (3.12) become identical with those for flow over a semi-infinite heated flat plate, and their numerical solution provides an initial solution for step-by-step integration of the full equations. To advance the solution beyond $X = 0$ in such a manner we proceed as follows. First we quasi-linearise (3.10). Then, with the solution known at $X = X_i$, X_{i-1} etc., we provide an estimate of U, Θ at $X = X_{i+1}$ by extrapolation from previous stations, and gain a first estimate of V from the solution of (3.9). The quasi-linearised equation (3.10) then provides an updated solution for U. In obtaining this we do not seek a fully converged, but only a partially converged, solution of (3.10). Finally we update our estimate of Θ by solving (3.11). This iterative scheme, through equations (3.9) to (3.11), is repeated until overall convergence according to some pre-set criterion is achieved. The method of solution adopted is a finite-difference method in which all derivatives in (3.9) to (3.11), and the heat-flow condition in (3.12), are approximated by central differences. Thus, second-order accuracy is achieved, and the solution of (3.10), (3.11) is simply an adaptation of the well-known Crank–Nicolson method. For the results presented in this paper we have set step lengths $\delta X = 0.05$, $\delta Y = 0.1$, and placed the outer boundary of the computational domain at $Y = Y_\infty = 60$. These values provide the necessary level of accuracy for our purposes.

A feature which complicates the procedure described above is the introduction, in the transformations (3.6) and (3.7), of the parameter X_0 which is unknown *a priori*. In order to implement our step-by-step procedure an estimate, X_0^e say, of this quantity must be made. If our estimate $X_0^e > X_0$ then, assuming the Kuiken solution is indeed appropriate, the solution will clearly exhibit a singular behaviour with, *inter alia*, $\Theta(X, 0) \to \infty$ as X approaches the unknown value X_0. On the other hand, if $X_0^e < X_0$, T remains finite as $X \to X_0^e$ and as a consequence, as we see from (3.7c), Θ and indeed $\partial^2\Theta/\partial X^2$ both approach zero. When X_0 is identified correctly Θ remains finite and smooth as X approaches X_0. With these considerations in mind it is not difficult to estimate X_0, and for the step length $\delta X = 0.05$ we readily find that $3.55 < X_0 < 3.60$. Proceeding in this way, with smaller step lengths δX, would enable us to refine this estimate. However, a consequence of the procedure described above is that Kuiken's solution is indeed emerging as a candidate for the limiting solution as $X \to X_0$. We therefore exploit that as follows. If we define a skin friction coefficient $C_f = \partial u/\partial y|_{y=0}$, then from (3.1), (3.2), (3.4) and (3.5) we have, close to $x = x_0$,

$$\left.\begin{aligned} T^{-1/3}(x, 0) &= 0.163075(x_0 - x) + O\{(x_0 - x)^2\}\,, \\ C_f^{-1/2}(x) &= 0.156136(x_0 - x) + O\{(x_0 - x)^2\}\,. \end{aligned}\right\} \tag{3.13}$$

If now, in the course of our numerical solution of (3.9) to (3.11), we tabulate the quantities shown in, and make a comparison with, equations (3.13) we can without difficulty estimate X_0 and hence x_0 to a high degree of accuracy. In this way we find

$$X_0 = 3.58634 \quad \text{and} \quad x_0 = 4.11715\,. \tag{3.14}$$

In Figs 2a and 2b we compare (3.13) with the corresponding quantities obtained from our numerical solution, with x_0 as in (3.14). The agreement is excellent and provides striking confirmation that Kuiken's solution is indeed the appropriate limiting solution as $x \to x_0$. To substantiate this further we shown in Figs 3a and 3b temperature and velocity profiles to demonstrate that these do indeed approach the limiting forms appropriate to Kuiken's solution.

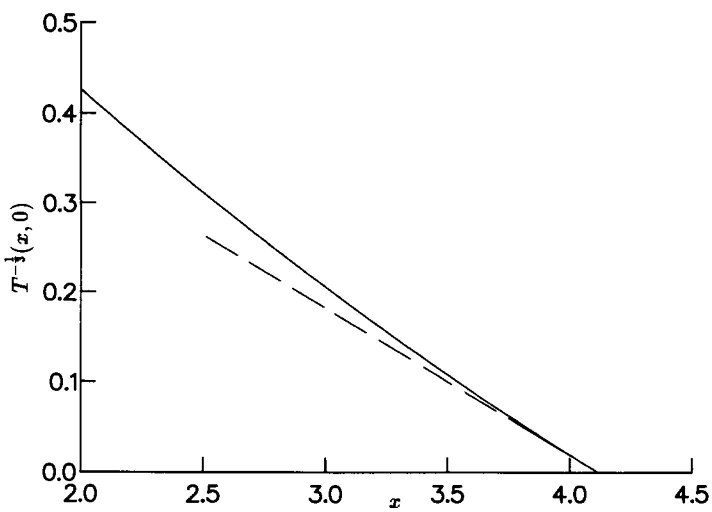

Fig. 2(a). A comparison between the wall temperature obtained from the full solution and the asymptotic result (3.13), shown as a broken line.

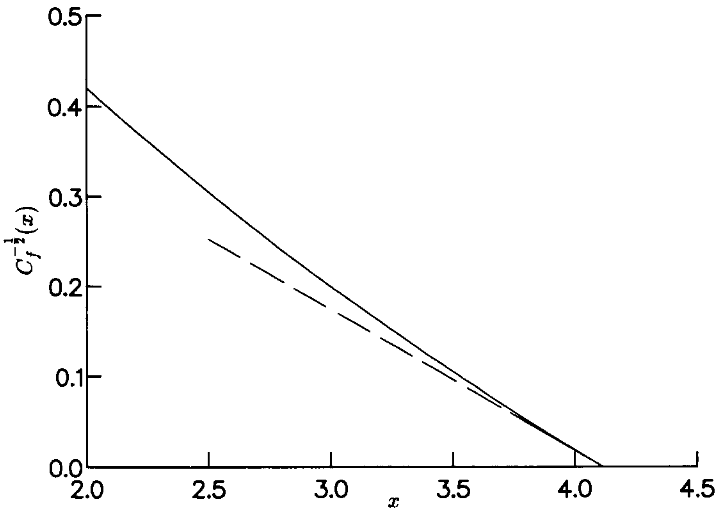

Fig. 2(b). A comparison between the skin friction coefficient C_f from the full solution and the asymptotic result (3.13), shown as a broken line.

4. Discussion

In the foregoing we have demonstrated that the singular, self-similar solution introduced by Kuiken [3], in association with the cooling of a vertically moving thin film has an important role to play in the solution of the axisymmetric boundary-layer equations for a hot, vertically downward moving fibre that is cooling. We have modelled the fibre as a long, thin cylinder of circular cross-section and shown that Kuiken's solution provides the leading term in the development of a singular solution in the neighborhood of the point at which the solution, of the boundary-layer equations, breaks down. A physical explanation for this breakdown is as

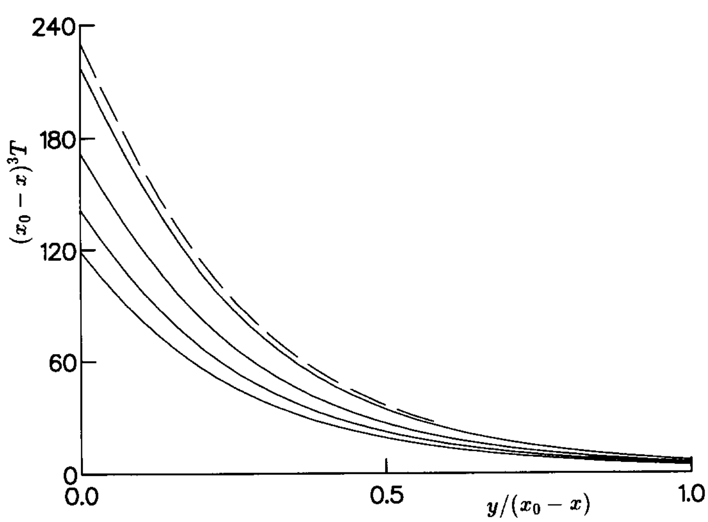

Fig. 3(a). Temperature profiles, successively as we move up, for $X = 2, 2.5, 3.0, 3.5$. The limit solution of Kuiken is shown as a broken line.

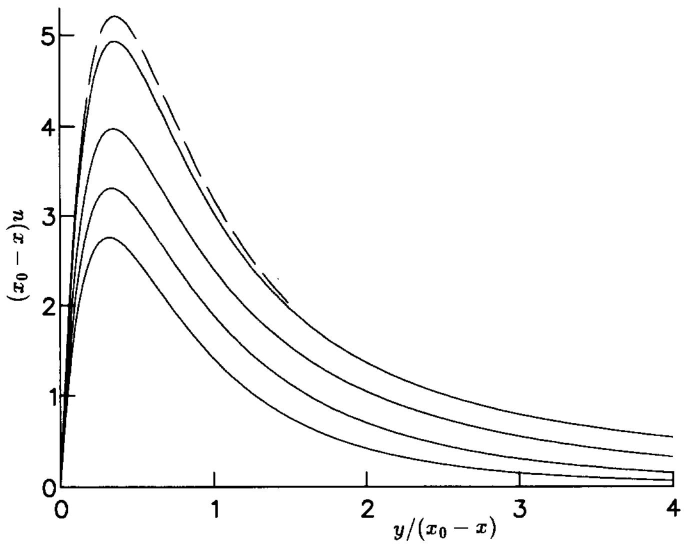

Fig. 3(b). Velocity profiles, successively as we move up, for $X = 2$, 2.5, 3.0, 3.5. The limit solution of Kuiken is shown as a broken line.

follows. Heat is carried down in the fibre from $x' = L$, and simultaneously free convective effects transport heat upwards. There is therefore an accumulation of heat which results in an acceleration of the free-convective process to the extent that the boundary-layer equations fail at some point $x' = x_0'$. As we see from equation (3.1), not only do the streamwise velocity and temperature become unbounded but so, also, does streamwise diffusion of vorticity and heat. In that case the boundary-layer equations become inappropriate to describe the flow. Whether or not such a breakdown does occur, leading to enhanced convection and therefore augmented cooling of the fibre, depends upon the length L of the exposed fibre. It is self-evident that the shorter the length of the exposed fibre the less effective is the cooling process. But what has been shown in the present investigation is that for a sufficient length of exposed fibre the free-convective, and hence cooling, process can be significantly enhanced.

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